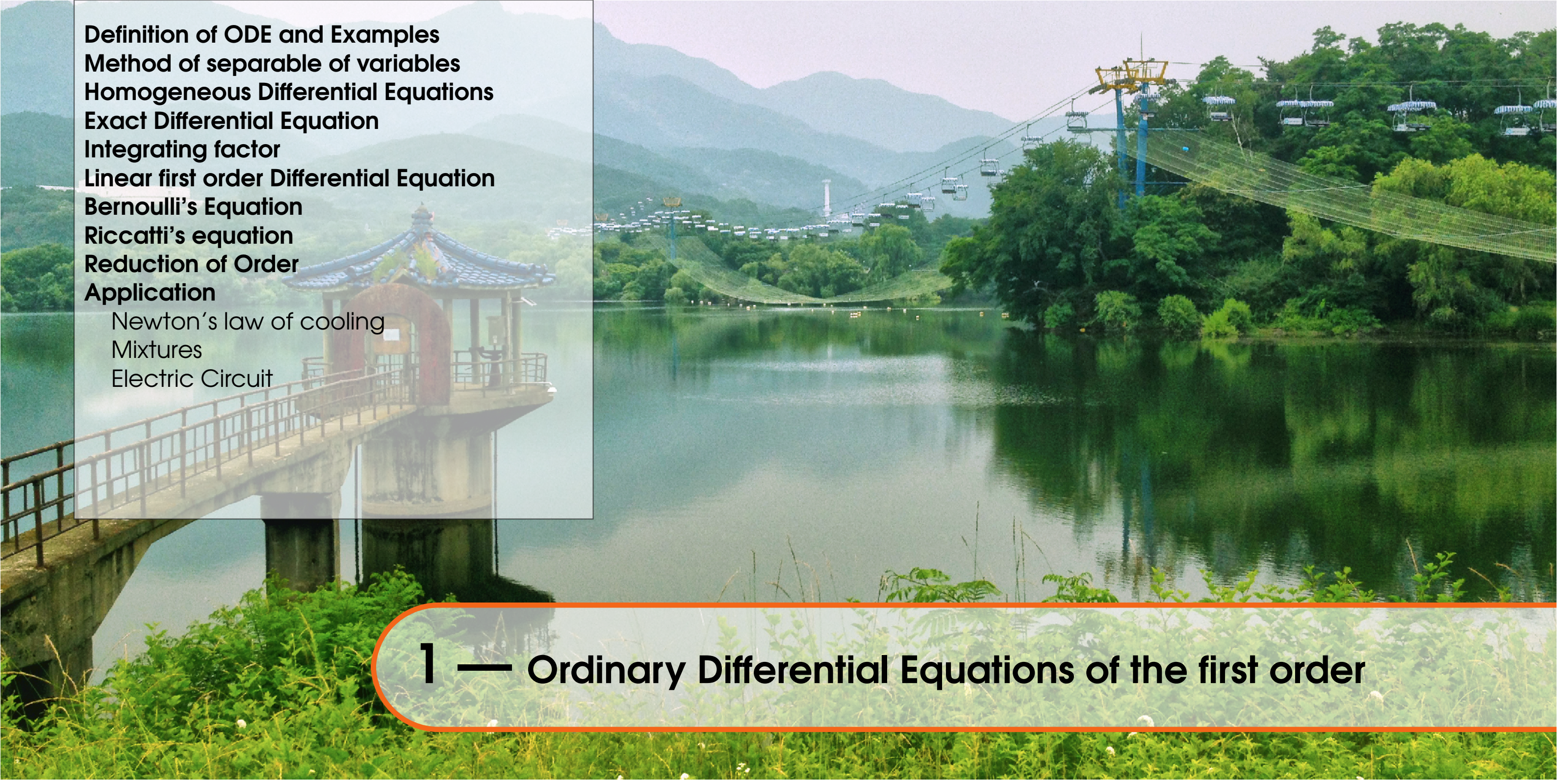

1 — Ordinary Differential Equations of the first order

1.1 Definition of ODE and Examples

Definition 1.1.1 A differential equation is an equation which involves derivatives.

Example 1.1

$$a)\ \frac{dy}{dx}=2x+5 \quad b)\ x\frac{d^2y}{dx^2}+6\frac{dy}{dx}=-2y \quad c)\ \frac{\partial z}{\partial x}-x\frac{\partial z}{\partial y}-z=0 \quad d)\ x\frac{dy}{dx}+y=3$$

$$e)\ \frac{d^3y}{dx^3}+2\left(\frac{d^2y}{dx^2}\right)^2+\frac{dy}{dx}=\cos x \quad f)\ \left(\frac{d^2y}{dx^2}\right)^2+\left(\frac{dy}{dx}\right)^4+9y=x^3 \quad g)\ \frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=x^2+y$$

■

A differential equation involving only ordinary derivatives (derivatives of functions of one variable) is called *ordinary differential equation*.

Example 1.2 Examples a, b, d, e, f are an example of ODE. ■

If there are two or more independent variables, the derivatives are partial derivatives and the equations are called a PDE.

Example 1.3 Examples c,and g are an example of PDE. ■

Order of a differential equation

Definition 1.1.2 The order of a differential equation is defined as the order of the highest derivative which appears in the equation.

Example 1.4 Examples a, c, d are of order one where as b, f, g are of order two and e is of order three ■

Definition 1.1.3 The degree of a differential equation is the highest power of the highest derivative in the equation, after the differential equation is expressed as a polynomial of the dependent variable and its derivatives.

Example 1.5

(a) $y'' - 2xy' + y = e^x$ order 2, degree 1

(b) $(y''')^4 + 5(y'')^5 - 2y' + y = x^2 + 2$ order 3, degree 4

(c) $(y')^{\frac{3}{2}} = y'' + 1$.
First let us write the given differential equation as a polynomial of the dependent variable and its derivatives, that is,
$(y')^{\frac{3}{2}} = y'' + 1 \implies (y')^3 = (y''+1)^2 = (y'')^2 + 2y'' + 1$.
Therefore, the degree of the given differential equation is 2.

Definition 1.1.4 A solution of ODE is free from derivatives and which satisfies the given differential equation.

If a solution of a differential equation is given explicitly as $y = f(x)$ we call it an explicit solution, otherwise it is of the form $h(x,y) = 0$ called implicit solution.

Example 1.6 Show that e^{2x} and e^{3x} are solution of $y'' - 5y' + 6y = 0$

Definition 1.1.5 A differential equation is said to be linear if it is linear in the dependent variable and its derivatives, and those coefficients are a function of the independent variable. *That is, a differential equation is linear if the independent variable and its derivatives are not multiplied together, not raised to powers, do not occur as the arguments of functions.*

A differential equation which is not linear in some dependent variable is said to be non linear.

R An n^{th} order differential equation is linear if it can be written of the form
$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x)$$
where the coefficients $a_i(x)$ are function of x alone.

Example 1.7 For example

	Differential equation	Linearity
1	$y'' + 4xy' + 2y = cosx$	Is linear, ordinary and order 2
2	$y'' + 4yy' + 2y = cosx$	Is nonlinear ($\because yy'$)
3	$\frac{\partial^2 u}{\partial x^2} + \frac{\partial v}{\partial t} + u + v = \sin u$	Is linear in v and nonlinear in u ($\because \sin u$). The equation is nonlinear.

Initial value problem and Boundary Value Problem

In application one may be interested to find a solution to a differential equation satisfying certain defined conditions and such conditions are called initial conditions. If all conditions are given at one point of independent variable the conditions are called initial conditions and if the conditions are given at more than one point of the independent variable the conditions are called boundary conditions.

Definition 1.1.6 An IVP is a problem which seeks to determine a solution to a differential equation on the unknown functions and its derivatives specified at one value of the independent variable.

Example 1.8 Consider the differential equation $\frac{d^2y}{dx^2} = x + 1$ subject to the condition $y(0) = 1, y'(0) = 0$. So the given problem is an IVP

Definition 1.1.7 A BVP is a problem which seeks to determine a solution to a differential equation subject to the boundary conditions on the unknown functions and its derivatives specified at least at two different values of the independent variable.

Example 1.9 Consider the differential equation $y'' + y = 0$ subject to the condition $y(0) = 0, y'(\frac{\pi}{2}) = 1$. So the given problem is BVP

1.2 Method of separable of variables

Definition 1.2.1 A differential equation of the form

$$g(x) + h(y)\frac{dy}{dx} = 0$$

is called separabel equation

The solution is obtained by integrating both sides with respaect to x

$$\int h(y)dy + \int g(x)dx = c$$

is the general solution.

Example 1.10 Solve the following differential equations by separation of variables

1. $\frac{dy}{dx} = \frac{x^2}{y}$
2. $\frac{dy}{dx} = \frac{x^2}{y(1+x^3)}$
3. $\frac{dy}{dx} = y - y^2$
4. $\frac{dy}{dx} = 1 + y^2 - 2x - 2xy^2, \quad y(0) = 0$

Solution:

1. The ODE $\frac{dy}{dx} = \frac{x^2}{y}$ becomes $ydy = x^2dx$

$$\Rightarrow \int ydy = \int x^2dx$$

$$\Rightarrow \frac{y^2}{2} = \frac{x^3}{3} + c$$

$$\Rightarrow y = \sqrt{\frac{2x^3}{3} + c}$$

2. The ODE $\dfrac{dy}{dx} = \dfrac{x^2}{y(1+x^3)}$ becomes $ydy = \dfrac{x^2}{1+x^3}dx$

$$\Rightarrow \int ydy = \int \frac{x^2}{1+x^3}dx$$
$$\Rightarrow \frac{y^2}{2} = \frac{1}{3}ln(1+x^3)+c$$
$$\Rightarrow y = \sqrt{\frac{2}{3}\ln(1+x^3)+c}$$

3. The ODE $\dfrac{dy}{dx} = y - y^2$ can be written as $\dfrac{1}{y-y^2}dy = dx$

$$\Longrightarrow \int \frac{1}{y-y^2}dy = \int dx \qquad \text{(Integrating both side w.r.t. } x\text{)}$$
$$\Longrightarrow \int \frac{1}{y(1-y)}dy = \int dx$$
$$\Longrightarrow \int \left(\frac{1}{y} - \frac{1}{y-1}\right)dy = \int dx \qquad \text{(integration by partial fraction)}$$
$$\Longrightarrow \ln(y) - \ln(y-1) = x + c$$
$$\Longrightarrow \ln\frac{y}{y-1} = x + c \Longrightarrow \frac{y}{y-1} = e^{x+c}$$
$$\Longrightarrow \frac{y}{y-1} = Ce^x \qquad \text{(where } C = e^c\text{)}$$
$$\Longrightarrow y = (y-1)Ce^x \Longrightarrow y = \frac{Ce^x}{Ce^x - 1}$$

4. The ODE can be written as $\dfrac{dy}{dx} = (1-2x) + y^2(1-2x)$

$$\Longrightarrow \frac{dy}{dx} = (1-2x)(1+y^2)$$
$$\Longrightarrow \frac{1}{1+y^2}dy = (1-2x)dx$$
$$\Longrightarrow \int \frac{1}{1+y^2}dy = \int (1-2x)dx \qquad \text{(integration by trignometric substitution let } y = \tan\theta\text{)}$$
$$\Longrightarrow \tan^{-1}y = x - x^2 + c$$
$$\Longrightarrow y = \tan(x - x^2 + c) \qquad \text{(The general solution)}$$

From the initial condition $y(0) = 0$, we obtain,

$$y(0) = \tan(0-0+c) \Longrightarrow 0 = \tan c \Longrightarrow c = \tan^{-1}0 = 0$$

Thus, the solution of the IVP is

$$y = \tan(x - x^2)$$

Exercise 1.1 Solve

(a) $y' + y^2 \sin x = 0$
(b) $y' = e^{x+y}$
(c) $\dfrac{2y}{y^2+1}\dfrac{dy}{dx} = \dfrac{1}{x^2}$
(d) $x^2y^2dx - (1+x^2)dy = 0, \quad y(0) = 1$
(e) $(e^{2y} - y)\cos x\dfrac{dy}{dx} = e^y \sin 2x, \quad y(0) = 0$
(f) $(xy + 2x + y + 2)dx + (x^2 + 2x)dy = 0$
(g) $(x\cos y + (x^2-1)\sin y\dfrac{dy}{dx} = 0, \quad y(0) = \frac{\pi}{3}$

■

1.3 Homogeneous Differential Equations

Definition 1.3.1 A function $f(x,y)$ is called homogeneous of degree n if $f(\lambda x,\lambda y)=\lambda^n f(x,y)$

Example 1.11 a $f(x,y)=x^5-x^2y^3$ is homogeneous of degree 5.

b $f(x,y)=x^3+\sin x\cos y$ is not homogeneous because $f(\lambda x,\lambda y)\neq\lambda^n f(x,y)$.

c $f(x,y)=e^{\frac{y}{x}}+\tan\dfrac{y}{x}$ is homogeneous of degree 0.

■

Definition 1.3.2 The differential equation of the form

$$M(x,y)dx+N(x,y)dy=0 \tag{1.1}$$

is said to be homogeneous iff $M(x,y)$ and $N(x,y)$ are homogeneous functions of the same degree.

Equation (1.1) can be written in the form

$$\frac{dy}{dx}=f(x,y) \text{ where } f(x,y)=-\frac{M(x,y)}{N(x,y)}$$

which is homogeneous of degree 0.

$$\Rightarrow\quad f(\lambda x,\lambda y)=\lambda^0 f(x,y)=f(x,y),\quad \text{set } \lambda=\frac{1}{x}$$

$$\Rightarrow\quad f(x,y)=f(1,\frac{y}{x})=f(1,z),\ \ z=\frac{y}{x}$$

$$\Rightarrow\quad y=zx\Rightarrow\frac{dy}{dx}=z+x\frac{dz}{dx}\quad\Rightarrow\quad z+x\frac{dz}{dx}=f(1,z)$$

$$\Rightarrow\quad \frac{xdz}{f(1,z)-z}=dx\quad\Rightarrow\quad\frac{dz}{f(1,z)-z}-\frac{dx}{x}=0\quad\text{(which is separable)}$$

Example 1.12 Solve

$$(y^2+2xy)dx-x^2dy=0$$

■

Solution: Both terms $(M(x,y)=y^2+2xy,\ \ \&\ \ N(x,y)=-x^2)$ in the differential equation are homogeneous of degree 2, so the equation itself is homogeneous. Differentiating the substitution $y=zx$ gives

$$\frac{dy}{dx}=z+x\frac{dz}{dx}\quad\Rightarrow\quad dy=zdx+xdz$$

The given differential equation $(y^2+2xy)dx-x^2dy=0$ becomes $\dfrac{dy}{dx}=\dfrac{y^2+2xy}{x^2}=\dfrac{y^2}{x^2}+\dfrac{2y}{x}$

Substituting $y=zx$ and $\dfrac{dy}{dx}$ in the differential equation we obtain the variables separable equation,

$$z+x\frac{dz}{dx}=z^2+2z\quad\Rightarrow\quad\frac{dz}{z^2+z}=\frac{dx}{x}\quad\Rightarrow\quad\int\frac{dz}{z^2+z}=\int\frac{dx}{x}$$

$$\Rightarrow\quad\int\frac{dz}{z(z+1)}=\ln x+\ln c\Rightarrow\ln z-\ln(z+1)=\ln cx$$

$$\Rightarrow\quad\ln\frac{z}{z+1}=\ln cx\quad\Rightarrow\quad\frac{z}{z+1}=cx\quad\Rightarrow\quad\frac{\frac{y}{x}}{\frac{y}{x}+1}\quad\Rightarrow\quad\frac{y}{x+y}=cx$$

Therefore the general solution of the given differential equation is

$$y=\frac{cx^2}{1-cx}$$

where C is an arbitrary constant. In this case the general solution is simple and y is determined explicitly in terms of x.

Example 1.13 Solve

$$y'=\frac{y-x}{y+x}$$

Solution: Let $f(x,y)=\dfrac{y-x}{y+x}$

$$f(\lambda x,\lambda y) = \frac{\lambda y-\lambda x}{\lambda y+\lambda x} = \frac{y-x}{y+x}=\lambda^0 f(x,y)$$

Thus, $f(x,y)$ is homogeneous function of degree 0, so the given differential equation itself is homogeneous. Differentiating the substitution y = zx gives

$$\frac{dy}{dx}=z+x\frac{dz}{dx} \quad\Rightarrow\quad dy=zdx+xdz$$

Substituting $y=zx$ and $\dfrac{dy}{dx}$ in the differential equation we obtain ,

$$\begin{aligned}
z+x\frac{dz}{dx}=\frac{zx-x}{zx+x} &\Longrightarrow z+x\frac{dz}{dx}=\frac{z-1}{z+1} \Longrightarrow x\frac{dz}{dx}=\frac{z-1}{1+z}-z\\
&\Longrightarrow x\frac{dz}{dx}=\frac{-z^2-1}{z+1}\\
&\Longrightarrow \frac{z+1}{-z^2-1}dz=\frac{1}{x}dx \Longrightarrow \int\frac{z+1}{-z^2-1}dz=\int\frac{dx}{x}\\
&\Longrightarrow \int\left(-\frac{z}{(z^2+1)}-\frac{1}{z^2+1}\right)dz=\int\frac{dx}{x}\\
&\Longrightarrow -\frac{1}{2}\ln(z^2+1)-\tan^{-1}(z)=\ln x+c\\
&\Longrightarrow \ln(z^2+1)+2\tan^{-1}(z)=-2\ln x+C\\
&\Longrightarrow \ln\left(\left(\frac{y}{x}\right)^2+1\right)+2\tan^{-1}\left(\frac{y}{x}\right)=-2\ln x+C\\
&\Longrightarrow \ln\left(\frac{x^2+y^2}{x^2}\right)+2\tan^{-1}\left(\frac{y}{x}\right)=-\ln x^2+C\\
&\Longrightarrow \ln(x^2+y^2)-\ln x^2+2\tan^{-1}\left(\frac{y}{x}\right)=-\ln x^2+C\\
&\Longrightarrow \ln(x^2+y^2)+2\tan^{-1}\left(\frac{y}{x}\right)=C \qquad \text{(implicit solution)}
\end{aligned}$$

Exercise 1.2 Solve

(a) $x^2ydx-(x^3+y^3)dy=0$

(b) $xy'=2x+3y$

(c) $\dfrac{dy}{dx}=\dfrac{y}{x}+\tan\dfrac{y}{x}$

(d) $(y^2-x^2)dx+xydy=0$

(e) $y^2dx-(x^2+xy)dy=0$

(f) $\dfrac{-1}{y}\sin\dfrac{x}{y}dx+\dfrac{x}{y^2}\sin\dfrac{x}{y}dy=0$

1.4 Exact Differential Equation

Definition 1.4.1 A differential equation of the form

$$M(x,y)dx+N(x,y)dy=0$$

is said to be exact if there is a function f such that $M(x,y)=\dfrac{\partial f}{\partial x}$ and $N(x,y)=\dfrac{\partial f}{\partial y}$

$$\begin{aligned}
&\Rightarrow \quad Mdx+Ndy=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy=0\\
&\Rightarrow \quad \frac{\partial f}{\partial x}+\frac{\partial f}{\partial}\frac{dy}{dx}=0 \quad \Rightarrow \quad \frac{d}{dx}f(x,y)=0\\
&\quad \therefore \quad f(x,y)=c \quad \text{is the general solution.}
\end{aligned}$$

R The partial derivatives of M and N exist and continuous.

Theorem 1.4.1 The differential equation

$$M(x,y)dx+N(x,y)dy=0$$

is exact if and only if $\dfrac{\partial N}{\partial x}=\dfrac{\partial M}{\partial y}$

Proof. ($\Longrightarrow$) Assume the given differential equation is exact. By definition, there exist a a differentiable function $f(x,y)$ such that

$$M(x,y)=\frac{\partial f}{\partial x} \qquad \text{and} \qquad N(x,y)=\frac{\partial f}{\partial y}$$

Therefore,

$$\frac{\partial M}{\partial y}=\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial N}{\partial x}$$

The equality of the mixed partials is a consequence of the continuity of the first partial derivatives of$M(x,y)$ and $N(x,y)$.

($\Longleftarrow$) To prove the converse of the theorem, we assume that $\dfrac{\partial N}{\partial x}=\dfrac{\partial M}{\partial y}$. We need to show that there is a function f such that

$$M(x,y)=\frac{\partial f}{\partial x} \qquad \text{and} \tag{1.2}$$

$$N(x,y)=\frac{\partial f}{\partial y} \tag{1.3}$$

Integrating equation (1.2) with respect to x, holding y fixed (this is a partial integration) to obtain

$$f(x,y)=\int M(x,y)dx+h(y) \tag{1.4}$$

where $h(y)$ is an arbitrary function of y (this is the integration "constant" that we must allow to depend on y, since we held y fixed in performing the integration).
Differentiating (1.4) partially with respect to y yields

$$N=\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}\int M(x,y)dx+\frac{dh}{dy}$$

$$\frac{dh}{dy}=N(x,y)-\frac{\partial}{\partial y}\int M(x,y)dx \tag{1.5}$$

To see that there is such a function of y, it suffices to show that the right-hand side in (1.5) is a function of y. We can find $g(y)$ by integrating with respect to y. Because the right hand side in (1.5) is defined on a rectangle, and hence on an interval as a function of x, it suffices to show that the derivative with respect to x is identically zero. But

$$\begin{aligned}
\frac{\partial}{\partial x}\left(N(x,y)-\frac{\partial}{\partial y}\int M(x,y)dx\right) &= \frac{\partial N}{\partial x}-\frac{\partial}{\partial x}\frac{\partial}{\partial y}\int M(x,y)dx \\
&= \frac{\partial N}{\partial x}-\frac{\partial}{\partial y}\frac{\partial}{\partial x}\int M(x,y)dx \\
&= \frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}=0
\end{aligned}$$

by hypothesis. so we can find desired function $h(y)$ by integrating eq. (1.5) with respect to y.

$$h(y)=\int\left(N(x,y)-\frac{\partial}{\partial y}\int M(x,y)dx\right)dy$$

Substituting this in Eq. (1.4), $f(x,y)$ becomes

$$f(x,y)=\int M(x,y)dx+\int\left(N(x,y)-\frac{\partial}{\partial y}\int M(x,y)dx\right)dy \tag{1.6}$$

as the desired function with $\dfrac{\partial f}{\partial x}=M$ and $\dfrac{\partial f}{\partial y}=N$ ■

Example 1.14 Test the differential equation

a $(9x^2+y-1)dx-(4y-x)dy=0$

b $e^y dx+(xe^y+2y)dy=0$

for exactness and solve it if it is exact.

Solution:a) $M(x,y)=9x^2+y-1,\;\; N(x,y)=-(4y-x)=x-4y$

$\Rightarrow\;\; \dfrac{\partial N}{\partial x}=1=\dfrac{\partial M}{\partial y}$

Hence the given differential equation is exact.

By definition there exist a function f of two variables such that

$$\frac{\partial f}{\partial x} = 9x^2+y-1 \tag{1.7}$$

$$\frac{\partial f}{\partial y} = x-4y \tag{1.8}$$

Integrating equation (1.7) with respect to x we get,

$$\begin{aligned}
f(x,y) &= \int(9x^2+y-1)dx \\
&= 3x^3+xy+h(y)
\end{aligned}$$

where h(y) is constant with respect to x. Differentiate w.r.t y, we obtain

$$\frac{\partial f(x,y)}{\partial y}=x+h'(y)$$

compainge this with equation (1.8), we have

$$x+h'(y)=x-4y \;\;\Rightarrow\;\; h'(y)=-4y \;\;\Rightarrow\;\; h(y)=-2y^2$$

$\therefore\;\; f(x,y)=3x^2+yx-x-2y^2=c$ is the general solution of the given ODE.

b) Ans. $f(x,y)=xe^y+y^2=c$

Example 1.15 Solve $(x+\tan^{-1}y)dx+\left(\dfrac{x+y}{1+y^2}\right)dy=0$

Solution: Here we have, $M(x,y)=x+\tan^{-1}y,\ \ N(x,y)=\dfrac{x+y}{1+y^2}$

$$\Rightarrow \frac{\partial N}{\partial x}=\frac{1}{1+y^2}=\frac{\partial M}{\partial y}$$

Hence the given differential equation is exact.
By definition there exist a function f of two variables such that

$$\frac{\partial f}{\partial x} = x+\tan^{-1}y \tag{1.9}$$
$$\frac{\partial f}{\partial y} = \frac{x+y}{1+y^2} \tag{1.10}$$

Integrating equation (1.9) with respect to x we get,

$$f(x,y) = \int(x+\tan^{-1}y)dx+h(y)=\frac{1}{2}x^2+x\tan^{-1}y+h(y)$$

where $h(y)$ is constant with respect to x. Differentiate w.r.t y, we obtain

$$\frac{\partial f(x,y)}{\partial y}=\frac{x}{1+y^2}+h'(y)$$

comparing this with equation (1.10), we get

$$\frac{x}{1+y^2}+h'(y)=\frac{x+y}{1+y^2} \implies h'(y)=\frac{y}{1+y^2}$$
$$\implies h(y)=\int\frac{y}{1+y^2}dy=\frac{1}{2}\ln(1+y^2)$$
$$\therefore\ f(x,y)=\frac{1}{2}x^2+x\tan^{-1}y+\frac{1}{2}\ln(1+y^2)$$

The general solution of the given ODE is

$$\frac{1}{2}x^2+x\tan^{-1}y+\frac{1}{2}\ln(1+y^2)=c$$

Exercise 1.3 Determine which of the following equations are exact and solve it, if it is exact

(a) $(e^x\sin y-2y\sin x)dx+(e^x\cos y+2\cos x)dy=0$
(b) $(y-x^3)dx+(x+y^3)dy=0$
(c) $(\sin x\sin y-xe^y)dy=(e^y+\cos x\cos y)dx$
(d) $dx=\dfrac{y}{1-x^2y^2}dx+\dfrac{x}{1-x^2y^2}dy$
(e) $\dfrac{-1}{y}\sin\dfrac{x}{y}dx+\dfrac{x}{y^2}\sin\dfrac{x}{y}dy=0$
(f) $(3x^2+y^2)dx+2xydy=0$
(g) $3x^2ydx+x^3dy=0$
(h) $2x\sin ydx+x^2\cos ydy=0$
(i) $(6xy-y^3)dx+(4y+3x^2-3xy^2)dy=0$

Answer: a) $e^x\sin y+2y\cos x=c$ b) $4xy-x^4+y^4=c$ c) $xe^y+\sin x\sin y=c$
d) $\ln\left(\dfrac{1+xy}{1-xy}\right)-2x=c$ e) $\cos\dfrac{x}{y}=c$ or $\dfrac{x}{y}=c$

1.5 Integrating factor

Definition 1.5.1 Let $M(x,y)dx+N(x,y)dy=0$ is not exact, any function μ which makes $\mu\,(M(x,y)dx+N(x,y)dy)=0$ exact is called integrating factor.

Example 1.16 Show that $\mu = ye^x$ is an integrating factor for the differential equation

$$\left(\frac{\sin y}{y}-2e^{-x}\sin x\right)dx+\left(\frac{\cos y+2e^{-x}\cos x}{y}\right)dy=0$$

and use this fact to find a solution to the differential equation.

Answer: $f(x,y)=e^x\sin y+2y\cos x$

Theorem 1.5.1 A differential equation of the form $M(x,y)dx+N(x,y)dy=0$ has an integrating factor if it has a general solution

Proof. Let $f(x,y)=c$ be the general solution. Then

$$\begin{aligned}
\frac{df}{dx}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\frac{dy}{dx}=0 \quad &\Rightarrow \quad \frac{dy}{dx}=-\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}\\
&\Rightarrow \quad \frac{dy}{dx}=-\frac{M}{N}=-\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}\\
&\Rightarrow \quad \frac{\frac{\partial f}{\partial y}}{N}=\frac{\frac{\partial f}{\partial x}}{M}=\mu \quad \text{(let)}\\
&\Rightarrow \quad \frac{\partial f}{\partial x}=\mu M \text{ and } \frac{\partial f}{\partial y}=\mu N
\end{aligned}$$

Multipling the given equation by μ

$$\mu M(x,y)dx+\mu N(x,y)dy=0$$

which is exact. ■

Finding the integrating factor

$$\begin{aligned}
&\mu M(x,y)dx+\mu N(x,y)dy=0\\
\Rightarrow\ &\frac{\partial(\mu N)}{\partial x}=\frac{\partial(\mu M)}{\partial y}\\
\Rightarrow\ &\mu\frac{\partial N}{\partial x}+N\frac{\partial \mu}{\partial x}=\mu\frac{\partial M}{\partial y}+M\frac{\partial \mu}{\partial y}\\
\Rightarrow\ &N\frac{\partial \mu}{\partial x}-M\frac{\partial \mu}{\partial y}=\mu\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)\\
\Rightarrow\ &\frac{1}{\mu}\left(N\frac{\partial \mu}{\partial x}-M\frac{\partial \mu}{\partial y}\right)=\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x} \qquad (1.11)
\end{aligned}$$

Case 1: Let μ be a function of alone then (1.11) become

$$\frac{1}{\mu}\left(N\frac{\partial \mu}{\partial x}\right) = \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}$$
$$\Rightarrow \quad \frac{1}{\mu}\frac{\partial \mu}{\partial x} = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = g(x) \quad \Rightarrow \quad \frac{d}{dx}(\ln \mu) = g(x)$$
$$\Rightarrow \quad \ln \mu = \int g(x)dx \quad \Rightarrow \quad \mu = e^{\int g(x)dx}$$

Case 2: Similarly if μ is a function of y alone

$$\frac{1}{\mu}\frac{\partial \mu}{\partial y} = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{-M} = h(y)$$
$$\Rightarrow \quad \mu = e^{\int h(y)dy}$$

Example 1.17 Solve $(3x^2y + 2xy + y^3)dx + (x^2 + y^2)dy = 0$

Solution: $M(x,y) = 3x^2y + 2xy + y^3, \quad N(x,y) = x^2 + y^2$

$\frac{\partial M}{\partial y} = 3x^2 + 2x + 3y^2 \qquad \frac{\partial N}{\partial y} = 2x$

The differential equation is not exact

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 3x^2 + 2x + 3y^2 - 2x = 3(x^2 + y^2)$$
$$\Rightarrow \quad g(x) = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = 3\frac{x^2 + y^2}{x^2 + y^2} = 3$$
$$\mu = e^{\int g(x)dx} = e^{\int 3dx} = e^{3x}$$

Thus the ODE is reduced in to

$$e^{3x}(3x^2y + 2xy + y^3)dx + e^{3x}(x^2 + y^2)dy = 0$$

which is exact.
By definition there exist a function f such that

$$\frac{\partial f}{\partial x} = e^{3x}(3x^2y + 2xy + y^3) \text{ and } \frac{\partial f}{\partial y} = e^{3x}(x^2 + y^2)$$

Integrating the second w.r.t. y,

$$f(x,y) = \int e^{3x}(x^2 + y^2)dy = e^{3x}(x^2y + \frac{y^3}{3} + c(x))$$
$$\Rightarrow \quad \frac{\partial f}{\partial x} = e^{3x}(2xy) + 3e^{3x}(x^2y + \frac{y^3}{3}) + c'(x)$$
$$= \quad e^{3x}(2xy + 3x^2y + y^3) + c'(x)$$

By comparing the above equation, we have $c'(x) = 0 \quad \Rightarrow \quad c(x) = c$. Hence

$$f(x,y) = e^{3x}(x^2y + \frac{y^3}{3}) = k$$

is the general solution.

Example 1.18 Solve the initial value problem $ydx+(2x-ye^{y})dy=0, \quad y(0)=1$

Solution: $M(x,y)=y, \quad N(x,y)=2x-ye^{y} \Rightarrow \dfrac{\partial M}{\partial y}=1, \quad \dfrac{\partial N}{\partial x}=2 \Rightarrow \dfrac{\partial M}{\partial y}\neq\dfrac{\partial N}{\partial x}$

The given differential equation is not exact.

$$\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}=1-2=-1$$

$$h(y)=\frac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{-M}=\frac{-1}{y}=\frac{1}{y}$$

$$\therefore \quad \mu=e^{\int h(y)dy}=e^{\int\frac{1}{y}dy}=e^{\ln y}=y$$

Thus the differential equation is reduced in to

$$y^2dx+y(2x-ye^{y})dy=0$$

which is exact.

By definition there exist a function f such that

$$\frac{\partial f}{\partial x}=y^2 \text{ and } \frac{\partial f}{\partial y}=y(2x-ye^2)$$

Integrating the first w.r.t. x, we get

$$f(x,y)=\int y^2dx=y^2x+g(y) \Longrightarrow \frac{\partial f}{\partial y}=2xy+g'(y)$$

Comparing with the second equation, we have

$$g'(y)=-y^2e^{y} \Longrightarrow g(y)=e^{y}(2+2y-y^2)$$
$$\Longrightarrow f(x,y)=xy^2+e^{y}(2+2y-y^2)$$

The general solution is $xy^2+e^{y}(2+2y-y^2)=c$

From the initial conditions, we get $c=3e$ and

$$xy^2+e^{y}(2+2y-y^2)=3e$$

is the solution of the IVP.

Example 1.19 Solve $ydx+3xdy=0$

Solution: $M(x,y)=y, \quad N(x,y)=3x \Longrightarrow \dfrac{\partial M}{\partial y}=1, \quad \dfrac{\partial N}{\partial y}=3$

The differential equation is not exact

$$\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}=1-3=-2$$

$$\Longrightarrow \quad g(x)=\frac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N}=\frac{-2}{3x}$$

$$\mu=e^{\int g(x)dx}=e^{\int\frac{-2}{3x}dx}=e^{\frac{-2}{3}\ln x}=\frac{1}{(x)^{2/3}}$$

OR

$$h(y) = \frac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{-M}=\frac{-2}{-y}=\frac{2}{y}$$

$$\mu = e^{\int h(y)dy}=e^{\int\frac{2}{y}dy}=e^{2\ln y}=y^2$$

Hence, for the given differential equation we have two integrating factor. This implies that, integrating factor is not unique.
Thus the ODE is reduced in to

$$\frac{y}{(x)^{2/3}}dx + 3(x)^{1/3}dy = 0 \quad \text{Or} \quad y^3dx + 3xy^2dy = 0$$

which is exact.
By definition there exist a function f such that

$$\frac{\partial f}{\partial x} = \frac{y}{(x)^{2/3}} \tag{1.12}$$

$$\frac{\partial f}{\partial y} = 3(x)^{1/3} \tag{1.13}$$

Integrating equation (1.13) w.r.t. y,

$$f(x,y) = \int 3(x)^{1/3}dy = 3yx^{\frac{1}{3}} + h(x)$$

$$\Rightarrow \quad \frac{\partial f}{\partial x} = \frac{y}{(x)^{2/3}} + h'(x)$$

By comparing the above equation with (1.12), we have $h'(x) = 0 \;\Rightarrow\; h(x) = c$. Hence

$$f(x,y) = 3yx^{\frac{1}{3}}$$

Hence, the general solution is given by

$$3yx^{\frac{1}{3}} = c \implies xy^3 = C$$

OR $\qquad y^3dx + 3xy^2dy = 0$

By definition there exist a function f such that

$$\frac{\partial f}{\partial x} = y^3 \tag{1.14}$$

$$\frac{\partial f}{\partial y} = 3xy^2 \tag{1.15}$$

Integrating equation (1.14) w.r.t. x,

$$f(x,y) = \int y^3dx = xy^3 + g(y)$$

$$\Rightarrow \quad \frac{\partial f}{\partial y} = 3y^2 + g'(y)$$

By comparing the above equation with (1.15), we have $g'(y) = 0 \;\Rightarrow\; g(y) = c$. Hence

$$f(x,y) = xy^3$$

Hence, the general solution is given by

$$xy^3 = C$$

Exercise 1.4 Solve the following differential equation by finding an integrating factor

(a) $(3xy + y^2) + (x^2 + xy)y' = 0$

(b) $(xy - 1)dx + (x^2 - xy)dy = 0$

(c) $y' + 2xy = e^{x - x^2}, \quad y(0) = -1$

1.6 Linear first order Differential Equation

The standard form of a first order differential equation is

$$\frac{dy}{dx} + p(x)y = q(x) \tag{1.16}$$

where p(x) and q(x) are any function of x.

$$\Rightarrow \quad (p(x)y - q(x))dx + dy = 0 \tag{1.17}$$

$$\Rightarrow \quad M(x,y) = p(x)y - q(x), \quad N(x,y) = 1$$

Equation (1.17) is not exact, exactness would require $M_y = N_x$

$$\frac{\partial M}{y} - \frac{\partial N}{x} = p(x)$$

$$\Rightarrow \quad \frac{\frac{\partial M}{y} - \frac{\partial N}{x}}{N} = \frac{p(x)}{1} = p(x)$$

$$\Rightarrow \quad \mu = e^{\int p(x)dx}$$

Multiplying equation (1.16) by μ

$$e^{\int p(x)dx}\frac{dy}{dx} + e^{\int p(x)dx}p(x) = e^{\int p(x)dx}q(x)$$

$$\Rightarrow \quad \frac{d}{dx}\left(ye^{\int p(x)dx}\right) = e^{\int p(x)dx}\frac{dy}{dx} + e^{\int p(x)dx}p(x) = e^{\int p(x)dx}q(x)$$

$$\Rightarrow \quad ye^{\int p(x)dx} = \int e^{\int p(x)dx}q(x)dx + c$$

$$\therefore \quad y = e^{-\int p(x)dx}\left[\int e^{\int p(x)dx}q(x)dx + c\right] \quad \text{(The general solution)}$$

Example 1.20 Solve the following differential equation

a $y' + y\cot x = \sin x$

b $y' + 2xy = 4x$

c $xy' - y = x^2e^{-x}$

■

Solution:

a $y' + y\cot x = \sin x$ is a linear first order differential equation with $p(x) = \cot x \& q(x) = \sin x$

$$\begin{aligned}
y &= e^{-\int p(x)dx}\left[\int e^{\int p(x)dx}q(x)dx + c\right] \\
&= e^{-\int \cot x dx}\left[\int e^{\int \cot x dx}\sin x dx + c\right] \\
&= e^{-\ln \sin x}\left[\int e^{\ln \sin x}\sin x dx + c\right] \\
&= \frac{1}{\sin x}\left[\int \sin^2 x dx + c\right] \\
&= \frac{1}{\sin x}\left[\int \frac{1-\cos 2x}{2}dx + c\right] \\
&= \frac{1}{\sin x}\left[\frac{1}{2x} - \frac{\sin 2x}{4} + c\right] \quad = \quad \frac{1}{\sin x}\left[\frac{1}{2x} - \frac{2\sin x\cos x}{4} + c\right] \\
&= \frac{x}{2\sin x} - \frac{\cos x}{2} + \frac{c}{\sin x}
\end{aligned}$$

b $\dfrac{dy}{dx}+2xy=4x,\quad p(x)=2x,\quad q(x)=4x$

$$\begin{aligned} y &= e^{-\int p(x)dx}\left[\int e^{\int p(x)dx}q(x)dx+c\right] \\ &= e^{-\int 2xdx}\left[\int e^{\int 2xdx}4xdx+c\right] \;=\; e^{-x^2}\left[\int e^{x^2}4xdx+c\right] \\ &= e^{-x^2}\left[2e^{x^2}+c\right]=2+ce^{-x^2} \end{aligned}$$

c $xy'-y=x^2e^{-x} \;\Rightarrow\; y'-\dfrac{1}{x}=xe^{-x},\quad p(x)=-\dfrac{1}{x},\quad q(x)=xe^{-x}$

$$\begin{aligned} y &= e^{-\int -\frac{1}{x}dx}\left[\int e^{\int -\frac{1}{x}dx}xe^{-x}dx+c\right] \\ &= e^{\ln x}\left[\int e^{-\ln x}xe^{-x}dx+c\right] \;=\; x\left[\int e^{-x}dx+c\right] \\ &= x[-e^{-x}+c]=-xe^{-x}+cx \end{aligned}$$

1.7 Bernoulli's Equation

The differential equation

$$\frac{dy}{dx}+p(x)y=q(x)y^n \tag{1.18}$$

where n is any real number, is called **Bernoulli's equation**. Note that for $n=0$ and $n=1$, equation (1.18) is linear. For $n\neq 0$ and $n\neq 1$ the substitution $z=y^{1-n}$ reduces any equation of form (1.18) to a linear equation.

$$\Longrightarrow\quad \frac{dz}{dx}=(1-n)y^{-n}\frac{dy}{dx}\quad\Longrightarrow\quad \frac{dy}{dx}=\frac{y^n}{(1-n)}\frac{dz}{dx}$$

Substitute in equation (**??**), we get

$$\begin{aligned} &\Longrightarrow\quad \frac{y^n}{(1-n)}\frac{dz}{dx}+p(x)y=q(x)y^n \\ &\Longrightarrow\quad \frac{1}{(1-n)}\frac{dz}{dx}+p(x)y^{1-n}=q(x)\quad \text{(Divide both sides by } y^n\text{)} \\ &\Longrightarrow\quad \frac{1}{(1-n)}\frac{dz}{dx}+p(x)z=q(x) \\ &\Longrightarrow\quad \frac{dz}{dx}+(1-n)p(x)z=(1-n)q(x)\quad \text{(Linear first order differential equation)} \end{aligned}$$

The general solution of the Bernoulli equation is

$$y^{1-n}e^{\int(1-n)p(x)dx}=\int(1-n)q(x)e^{\int(1-n)p(x)dx}dx+c$$

Example 1.21 Solve $x\dfrac{dy}{dx}+y=x^2y^2$ ■

Solution: We first rewrite the equation as

$$\frac{dy}{dx}+\frac{1}{x}y=xy^2$$

by dividing by x. With $n=2$ we have $z=y^{-1}$. We then substitute

$$\frac{dy}{dx}=-y^2\frac{dz}{dx}$$

into the given equation and simplify. The result is

$$\begin{aligned} & -y^2\frac{dz}{dx}+\frac{1}{x}y=xy^2 \implies \frac{dz}{dx}-\frac{1}{x}y^{-1}=-x \\ \implies & \frac{dz}{dx}-\frac{1}{x}z=-x \implies P(x)=-\frac{1}{x}, Q(x)=-x \end{aligned}$$

Hence,

$$\begin{aligned} z &= e^{-\int P(x)dx}\left[\int e^{\int P(x)dx}Q(x)dx+c\right] \\ &= e^{-\int -\frac{1}{x}dx}\left[\int e^{\int -\frac{1}{x}dx}(-x)dx+c\right]=e^{\ln x}\left[\int -xe^{-\ln x}dx+c\right] \\ &= x\left[-\int dx+c\right]=x[-x+c]=-x^2+cx \\ \implies y^{-1} &= -x^2+cx \\ \implies y &= \frac{1}{-x^2+cx} \end{aligned}$$

Exercise 1.5 Solve

(a) $x\frac{dy}{dx}+y=x^4y^3$ (b) $3y^2\frac{dy}{dx}+xy^3=x$ (c) $y'+xy=xe^{-x^2}y^{-3}$

1.8 Riccatti's equation

A differential equation of the form

$$\frac{dy}{dx}+p(x)y+q(x)y^2=r(x) \tag{1.19}$$

is a Riccatti differential equation.

- If $q(x)=0$, then (1.19) is first order linear differential equation.
- If $r(x)=0$, then (1.19) is Bernoulli's differential equation

Riccati differential equation can be solved if at least one non-trivial particular solution is known.

Suppose that $u=u(x)$ is a solution of (1.19) and make the change of variables $y=u+v$ to reduce the Riccati equation into Bernoulli equation. Then $y'=u'+v'$ and the differential equation (1.19) becomes

$$\begin{aligned} & u'+v'+p(x)(u+v)+q(x)(u+v)^2=r(x) \\ \implies & u'+v'+p(x)(u+v)+q(x)\left(u^2+2uv+v^2\right)=r(x) \\ \implies & u'+p(x)u+q(x)u^2+v'+p(x)v+q(x)v^2+2uvq(x)=r(x) \\ \implies & v'+(p(x)+2q(x)u)v+q(x)v^2=0 \quad \text{Since } u'+p(x)u+q(x)u^2=r(x) \\ \implies & v'+(p(x)+2q(x)u)v=-q(x)v^2 \quad \textbf{(Bernoulli's differential equation)} \end{aligned}$$

■ **Example 1.22** Solve $y' + \frac{1}{x}y - y^2 = -\frac{4}{x^2}$ with $u = \frac{2}{x}$ a given solution. ■

Solution: The given differential equation is Riccati's differential equation.

Let $y = \frac{2}{x} + v$ is the general solution. Then $y' = -\frac{2}{x^2} + v'$. Substituting into the given differential equation:

$$
\begin{aligned}
&-\frac{2}{x^2} + v' + \frac{1}{x}\left(\frac{2}{x} + v\right) - \left(\frac{2}{x} + v\right)^2 = -\frac{4}{x^2} \\
&-\frac{2}{x^2} + v' + \frac{2}{x^2} + \frac{1}{x}v - \frac{4}{x^2} - \frac{4}{x}v - v^2 = -\frac{4}{x^2} \\
&v' - \frac{3}{x}v = v^2 \qquad \text{Bernoulli equation with } n = 2
\end{aligned}
$$

Let $z = v^{-1}$. Then, $\frac{dz}{dx} = -v^{-2}\frac{dv}{dx} \implies \frac{dv}{dx} = -v^2\frac{dz}{dx}$

Substituting:

$$
\begin{aligned}
& v' - \frac{3}{x}v = v^2 \implies -v^2\frac{dz}{dx} - \frac{3}{x}v = v^2 \\
& \frac{dz}{dx} + \frac{3}{x}\frac{1}{v} = -1 \implies \frac{dz}{dx} + \frac{3}{x}z = -1 \\
\implies \quad & z = e^{-\int \frac{3}{x}dx}\left[\int e^{\int \frac{3}{x}dx}(-1)dx + c\right] \\
\implies \quad & \quad = e^{-3\ln x}\left[-\int e^{3\ln x}dx + c\right] \\
\implies \quad & v^{-1} = \frac{1}{x^3}\left[-\int x^3 + c\right] = \frac{1}{x^3}\left[\frac{1}{4}x^4 + c\right] \\
\implies \quad & \frac{1}{v} = \frac{1}{4}x + \frac{c}{x^3} = \frac{x^4 + c_1}{4x^3} \\
\implies \quad & v = \frac{4x^3}{x^4 + c_1}
\end{aligned}
$$

Therefore, the general solution for the given Riccati equation is

$$y = \frac{2}{x} + \frac{4x^3}{x^4 + c_1}$$

1.9 Reduction of Order

Some differential equation of the second order can be solved by reducing to a first order differential equation.

The general second order differential equation has the from

$$F(x,\, y,\, y',\, y'') = 0$$

To solve we consider two special cases

i **Dependent variable missing**

$$f(x,\, y',\, y'') = 0$$

Let $y' = p$ and $y'' = \frac{dp}{dx}$. Then

$$f(x, p, \frac{dp}{dx}) = 0 \to \text{(reduced to first order ODE in } p)$$

Example 1.23 Solve $xy'' - y' = 3x^2$

Solution: The differential equation reduced to

$$x\frac{dp}{dx} - p = 3x^2 \implies \frac{dp}{dx} - \frac{1}{x}p = 3x$$
$$\implies p = y' = 3x^2 + c_1 x$$
$$\implies y = x^3 + \frac{1}{2}c_1 x^2 + c_2$$

ii **Independent variable missing**

$$g(y,\ y',\ y'') = 0$$

Let $y' = p$ and $y'' = \frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = p\frac{dp}{dy}$. Then

$$g(y, p, p\frac{dp}{dy}) = 0 \to \text{(reduced to first order ODE in p)}$$

Example 1.24 Solve $2yy'' - (y')^2 = 1$

Solution: The given differential equation reduced to

$$2yp\frac{dp}{dy} - p^2 = 1 \implies 2yp\frac{dp}{dy} = p^2 + 1$$
$$\implies \frac{2p}{p^2+1}dp = \frac{1}{y}dy \implies \int \frac{2p}{p^2+1}dp = \int \frac{1}{y}dy$$
$$\implies p = \sqrt{c_1 y - 1} = \frac{dy}{dx} \implies dx = \frac{1}{\sqrt{c_1 y - 1}}dy$$
$$\implies y = \frac{1}{2}c_1 x\sqrt{c_1 y - 1} + c$$

Example 1.25 Solve $y'' + k^2 y = 0$ k is constant

Solution: The differential equation reduced to

$$p\frac{dp}{dy} + k^2 y = 0 \implies pdp + k^2 ydy = 0$$
$$\implies p^2 + k^2 y^2 = k^2 a^2 \implies p = y' = \pm k\sqrt{a^2 - y^2}$$
$$\implies \frac{dy}{\sqrt{a^2 - y^2}} = \pm kdx \implies \sin^{-1}\frac{y}{a} = \pm kx + b$$
$$\implies y = a\sin(\pm kx + b)$$

The general solution can be $y = c_1 \sin kx + c_2 \cos kx$ (by expanding $\sin(kx + B)$ & changing the from of constant)

Exercise 1.6 Solve the following

(a) $yy'' + (y')^2 = 0$ (b) $y'' - k^2 y = 0$ (c) $(x^2 + 2y')y'' + 2xy' = 0,\ y(0) = 1,\ y'(0) = 0$
d $xy'' = y' + (y')^3$, (e) $yy'' = y^2 y' + (y')^2,\ y(0) = \frac{-1}{2},\ y'(0) = 1$

1.10 Application

1.10.1 Newton's law of cooling

According to Newton's empirical law of cooling, the rate at which the temperature of a body changes is proportional to the difference between the temperature of the body and the temperature of the surrounding medium. If $T(t)$ represents the temperature of a body at time t, T_m the temperature of the surrounding medium, and $\frac{dT}{dt}$ the rate at which the temperature of the body changes, then

$$\frac{dT}{dt} \propto T - T_m \quad or \quad \frac{dT}{dt} = k(T - T_m)$$

where k is a constant of proportionality.

Example 1.26 A pot of liquid is put on the stove to boil. The temperature of the liquid reaches 170^0F and then the pot is taken off the burner and placed on a counter in the kitchen. The temperature of the air in the kitchen is 76^0F. After two minutes the temperature of the liquid in the pot is 123^0F. How long before the temperature of the liquid in the pot will be 84^0F?

Solution: $T_m = 76,\ \frac{dT}{dt} = k(T-76),\ T(0) = 170.$
Solving this separable differential equation, we get

$$\begin{aligned}
&\frac{dT}{T-76} = kdt \;\Rightarrow\; \int \frac{dT}{T-76} = \int kdt \;\Rightarrow\; \ln(T-76) = kt + c_1\\
\Rightarrow\;& T - 76 = Ce^{kt} \;\Rightarrow\; T(t) = Ce^{kt} + 76 \;\Rightarrow\; T(0) = 170 \;\Rightarrow\; C = 170 - 76 = 94\\
& T(2) = 123 \;\Rightarrow\; 123 = 94e^{2k} + 76 \;\Rightarrow\; 47 = 94^{2k} \;\Rightarrow\; k = \frac{1}{2}\ln\frac{1}{2} = -0.3466\\
\therefore\;& T(t) = 94e^{-0.3466t} + 76\\
\Rightarrow\;& 84 = 94e^{-0.3466t} + 76 \;\Rightarrow\; 8 = 94e^{-0.3466t} \;\Rightarrow\; -0.3466t = -2.4639 \;\Rightarrow\; t = 7.1088
\end{aligned}$$

When $t = 7.1088$ minutes the temperature of the liquid in the pot is 84^0F

Exercise 1.7

1. An object with temperature 150^0c is placed in a freezer whose temperature is 30^0c assume that newtons law of cooling applies and that the temperature of the freezer remains essentially constant. If this object is cooled to 120^0c after 8 minutes, what will its temperature be after 16 minutes? When will its temperature be 60^0c?
2. A thermometer is removed from a room where the temperature is 70^oF and is taken outside, where the air temperature is 10^oF. After one-half minute the thermometer reads 50^oF. What is the reading of the thermometer at $t = 1$ min? How long will it take for the thermometer to reach 15^oF?
3. The rate at which a body loses temperature at any instant is proportional to the amount by which the temperature of the body exceeds room temperature at the instant. A container of hot liquid is placed in a room of temperature 19^0c and in 8 minutes the liquid cools from 83^0c to 51^0c. How long does it takes for the liquid to cool from 27^0c to 25^0c?

1.10.2 Mixtures

Mixing problem occur quite frequently in chemical industry. Mixture problems generally concern a tank, or reservoir, containing a solution of some substance, being filled at a certain rate with another solution of the same substance, instantaneously mixed with the solution in the tank, and at the same time being drained at a certain rate.
The mixing of two salt solutions of differing concentrations gives rise to a first-order differential equation for the amount of salt contained in the mixture.
Let $A(t)$ denotes the amount of substance in the tank at time t, then the rate at which $A(t)$ changes is a net rate:

$$\frac{dA}{dt} = (\text{input rate of salt}) - (\text{output rate of salt}) = R_{in} - R_{out}$$

$$\begin{aligned} \text{where } R_{in} &= (\text{Flow rate of the liquid entering })(\text{Concentration of salt in it}) \\ R_{out} &= (\text{Flow rate of the liquid leaving})(\text{Concentration of salt in it}) \end{aligned}$$

$$\text{Concentration of salt in the tank at any time t} = \frac{A(t)}{\text{volume of fluid in the tank at any time}}$$

But the volume of brine at time t is given by

$$\begin{aligned} &(\text{initial volume}) + (\text{net change in volume}) \\ &= (\text{initial volume}) + (\text{flow rate entering} - \text{flow rate exit})t \end{aligned}$$

Example 1.27 A large tank holds 300 gallons of brine solution. Salt was entering and leaving the tank;A concentration of 2 lbs/gal is pumped into the tank at a rate of 3 gal/min; it mixed with the solution there, and then the mixture was pumped out at the rate of 3 gal/min. If 50 pounds of salt were dissolved initially in the 300 gallons, how much salt is in the tank after a long time? ■

Solution: $\frac{dA}{dt} = R_{in} - R_{out}$

$$R_{in} = \left(2\frac{lbs}{gal}\right)\left(3\frac{gal}{min}\right) = 6\frac{lbs}{min}$$

Now, since the solution is being pumped out of the tank at the same rate that it is pumped in, the number of gallons of brine in the tank at time t is a constant 300 gallons. Hence the concentration of the salt in the tank as well as in the outflow is $c(t) = \frac{A(t)}{300\ lb/gal}$

$$R_{out} = \left(\frac{A}{300}\frac{lbs}{gal}\right)\left(3\frac{gal}{min}\right) = \frac{A}{100}\frac{lbs}{min}$$

$$\begin{aligned} \frac{dA}{dt} &= 6 - \frac{A}{100}, \quad A(0) = 50 \\ \frac{dA}{dt} &+ \frac{A}{100} = 6, \quad p(t) = \frac{1}{100}, \quad q(t) = 6 \end{aligned}$$

$$\begin{aligned} A(t) &= 600 + ce^{-t/100}, \quad A(0) = 50, \\ \Rightarrow \quad 600 + ce^{-0/100} &= 50 \quad \Rightarrow \quad c = -550 \times 10^7 \end{aligned}$$

Thus the amount of salt in the tank at time t is given by

$$A(t) = 600 - 550e^{-t/100}$$

over a long time the number of pounds of salt in the solution must be 600 lb

Example 1.28 A large tank holds 300 gallons of brine solution with 40 lbs of salt. A concentration of 2 lbs/gal is pumped in at a rate of 4 gal/min. The concentration leaving the tank is pumped out at a rate of 3 gal/min. How much salt is in the tank after 12 min? ■

Solution: $\frac{dA}{dt} = R_{in} - R_{out}$

$$\begin{aligned} R_{in} &= \left(2\frac{lbs}{gal}\right)\left(4\frac{gal}{min}\right) = 8\frac{lbs}{min} \\ R_{out} &= \left(\frac{A}{300+t}\frac{lbs}{gal}\right)\left(3\frac{gal}{min}\right) = \frac{3A}{300+t}\frac{lbs}{min} \end{aligned}$$

$$\begin{aligned} \frac{dA}{dt} &= 8 - \frac{3A}{300+t}, \quad A(0) = 40 \\ \frac{dA}{dt} &+ \frac{3}{300+t}A = 8, \quad p(t) = \frac{3}{300+t}, \quad q(t) = 8 \end{aligned}$$

$$\begin{aligned} A(t) &= 600 + 2t + \frac{c}{(300+t)^3}, \quad A(0) = 40, \\ \Rightarrow \quad 600 + \frac{c}{300^3} &= 40 \quad \Rightarrow \quad c = -1512 \times 10^7 \end{aligned}$$

How much salt is in the tank after 12 min?

$$A(12) = 600 + 2(10) - \frac{1512 \times 10^7}{(300+12)^3} \approx 126.12\, lbs \quad \text{of salt}$$

Example 1.29 In an oil refinery, a storage tank contains 2000 gal of gasoline that initially has 100 lb of an additive dissolved in it. In preparation for winter weather, gasoline containing 2 lb of additive per gallon is pumped into the tank at a rate of 40 gal/min. The well-mixed solution is pumped out at a rate of 45 gal/min. How much of the additive is in the tank 20 min after the pumping process begins? ■

Solution: Let A be the amount (in pounds) of additive in the tank at time t. We know that $A = 100$ when $t = 0$. The number of gallons of gasoline and additive in solution in the tank at any time t is

$$V(t) = 200 + (40\, gal/min - 45\, gal/min)(t\, min) \; = \; (2000 - 5t)\, gal$$

Therefore, $R_{out} = \frac{A(t)}{V(t)} \times (Rate\ out\ flow) \; = \; \left(\frac{A(t)}{2000-5t}\right)45,$

$R_{in} = \left(2\,\frac{lb}{gal}\right)\left(40\,\frac{gal}{min}\right) \; = \; 80\,\frac{lb}{min}$

The differential equation modeling the mixture process is

$$\frac{dA}{dt} = R_{in} - R_{out} = 80 - \frac{45A}{2000-5t}$$

in pounds per minute.

Thus the general solution is

$$\begin{aligned} A = 2(2000-5t) + C(2000-5t)^9, \quad A(0) = 100 \; \Rightarrow \; C = -\frac{3900}{(2000)^9} \\ \Rightarrow \; A(20) = 1342\, lb \end{aligned}$$

Exercise 1.8
1. Consider a large tank holding 1000 L of pure water into which a brine solution of salt begins to flow at a constant rate of 6 L/min. The solution inside the tank is kept well stirred, and is flowing out of the tank at a rate of 6 L/min. If the concentration of salt in the brine solution entering the tank is 0.1 Kg/L, determine when the concentration of salt will reach 0.05 Kg/L.
2. Consider a tank in which 1 g of chlorine is initially present in $100m^3$ of a solution of water and chlorine. A chlorine solution concentrated at $0.03g/m^3$ flows into the tank at a rate of $1m^3/min$, while the uniformly mixed solution exits the tank at $2m^3/min$. At what time is the maximum amount of chlorine present in the tank, and how much is present?

1.10.3 Electric Circuit

A **RL-Series circuit:** Kirchoff's second law states that the sum of the voltage, $V(t)$ drop across the inductor, $L(\frac{dI}{dt})$ and across the resistor RI is the same as the impressed voltage $V(t)$ in the circuit where I is current.
$$V(t) = RI + L\frac{dI}{dt}$$
B **RC-Series circuit:** Kirchoff's second law states that the sum of the voltage, $V(t)$ drop across the capacitor, $\frac{1}{C}q(t)$ and across the resistor RI is the same as the impressed voltage $V(t)$ in the circuit where q is the charge on the capacitor.
$$V(t) = RI + \frac{1}{c}q$$

Example 1.30 An RL-circuit has an electromotive force of 5 volts, a resistor of 50Ω an inductance of 1 Henry and no initial current. Find the current in the circuit at any time.

Solution: $V(t) = RI + L\frac{dI}{dt}$, $V = 5, \; R = 50, \; L = 1$

$50I + \frac{dI}{dt} = 5 \;\Rightarrow\; I(t) = \frac{1}{10} - \frac{1}{10}e^{-50t}$

Example 1.31 A 100-volt electromotive force is applied to an RC series circuit in which the resistance is 200 ohms and the capacitance is 10^{-4} farad. Find the charge $q(t)$ on the capacitor if $q(0) = 0$. Find the current i(t).

Solution: $V(t) = RI + \frac{1}{c}q \;\Longrightarrow\; V(t) = R\frac{dq}{dt} + \frac{1}{c}q$,
$V = 100, \; R = 200, \; C = 10^{-4}$
$$\Longrightarrow \quad 200\frac{dq}{dt} + \frac{1}{10^{-4}}q = 100 \quad \Longrightarrow \quad \frac{dq}{dt} + 50q = \frac{1}{2}$$
$$\Longrightarrow \quad q = e^{-\int 50dt}\left[\int \frac{1}{2}e^{\int 50dt}dt + c\right] \quad \Longrightarrow \quad q = e^{-50t}\left[\int \frac{1}{2}e^{50t}dt + c\right]$$
$$\Longrightarrow \quad q = e^{-50t}\left[\frac{1}{100}e^{50t} + c\right] \quad \Longrightarrow \quad q = ce^{-50t} + \frac{1}{100}$$
From the initial condition, $q(0) = 0$, we obtain $c = -\frac{1}{100}$. Thus,
$$q = \frac{1}{100} - \frac{1}{100}e^{-50t} \text{ and } I = \frac{dq}{dt} = \frac{1}{2}e^{-50t}$$